Spectra of baroclinic inertia-gravity wave turbulence

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ABSTRACT

Baroclinic inertia-gravity (IG) waves form a persistent background of thermocline depth and sea surface height (SSH) oscillations. Measured by the ratio of water particle velocity to wave phase speed (alternatively, of the thermocline oscillation amplitude to the mean depth of the thermocline), the nonlinearity of IG waves may be rather high. Given a continuous supply of energy from external sources, nonlinear wave-wave interactions among IG waves would result in inertial cascades of energy, momentum and, possibly, wave action. Based on a recentl y developed theory of wave turbulence in scale-dependent systems, these cascades are investigated and IG wave spectra are derived for an arbitrary degree of wave nonlinearity. Comparison with satellite-altirnetry-based spectra shows good agreement. Finally, we discuss a possibility of inferring the internal Rossby radius of deformation and other dynamical properties of the upper thermocline from the spectra of SSH variations based on altimeter measurements.

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1. Introduction

Possible mechanisms of inertia-gravity (IG) wave generation include tidal forcing, atmospheric pressure and wind stress fluctuations and various types of hydrodynamic instability of vertical oceanic motions. Due to these, virtually permanent, sources of energy, IG waves form a persistent background of ocean oscillations. However, with respect to large-scale ocean circulation, open-ocean IG waves (as opposed to coastal and equatorial trapped waves) are usually considered to be of little importance. For example, for numerical modeling, these waves, especially the fast barotropic modes, represent "computational noise" which has to be filtered out, e.g., by employing a rigid-lid condition at the surface. One factor that might increase importance of open-ocean IG waves is nonlinear wave-wave interactions resulting in spectral fluxes of energy (as well as action and momentum), hence in a broad spectrum of sea surface height (SSH) variations. Oceanographic implications of these fluxes are discussed in Section 5 Finally, IG wave turbulence offers a plausible explanation of the peculiar shape of SSH spectra known from satellite altimeter observations - as discussed in section 4.

In the present work, based on a heuristic analysis of nonlinear resonant interactions between IG waves, we derive spectral distributions of wave energy and surface height variations. As shown in section 4, wave turbulence can develop only in baroclinic modes (we use the Boussinesq approximation) and is typical] y rather strong. Predicted spectra are analyzed in section 4 where a comparison with satellite altimeter observations is provided.

In the range of scales containing a Rossby radius of deformation, IG waves are characterized by a rather complicated dispersion law:

$$\omega^2 = f^2 + C_{0,m}^2 k^2 \tag{1}$$

where $k = |\mathbf{k}|$, f is the Coriolis parameter (considered to be constant) and $C_{0,m}$ is the phase velocity of Kelvin waves for vertical mode number m. A special case of $m=0$

corresponds to barotropic waves with $C_{0,0} = \sqrt{gH}$ where H is the (constant) ocean depth. Short- and long-wave asymptotic of (1) are obtained, respectively, for wavenumbers much greater and much smaller than the. inverse Rossby radius

$$R^{-1} = \frac{f}{C_{0,m}} \qquad (2)$$

The presence of characteristic scale R in (1) makes the. problem of IG turbulence highly non-trivial. In the short-wave approximation and for the lowest degree of nonlinearity, the wave spectrum was derived [Falkovich and Medvedev, 1992] by taking advantage of an approximate scale-invariance of the collision integral. As shown in sections 3 and 4, these assumptions represent an oversimplification in the case of ocean waves. Therefore, we employ an alternative approach called the multiwave interaction theory. Developed originally for deep-water gravity and capillary-gravity waves [Glazman, 1992-1995], it does not require either scale-invariance or weak nonlinearity. However, the heuristic nature of this theory makes it difficult to estimate the range of its validity. The experimental data presented in section 4 seem to corroborate the theoretical predictions, although some issues remain unresolved. The most difficult one is the dependence of the effective number of the resonantly interacting wave components on the appropriate external factors controlling the degree of the wave nonlinearity. These are discussed in sections 3 and 5.

For the reader unfamiliar with the subject, a few relevant concepts on wave turbulence are sketched in the next section. This presentation is rather qualitative and is focused on physical ideas; a more rigorous and detailed account of this material is available in the special literature [Zakharov and L'vov, 1974; Zakharov, 1984; Zakharov et al., 1992].

2. IG Wave Turbulence: An Overview

We consider wave propagation only in the horizontal plane. The governing equations for either barotropic or baroclinic waves in a shallow rotating fluid are

$$\frac{d\mathbf{U}}{dt} + f\mathbf{k} \times \mathbf{U} = -g\nabla\zeta$$
$$\frac{d\zeta}{dt} + \nabla \cdot \left((H+\zeta)\mathbf{U} \right) = 0 \qquad (3)$$
$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

Here, U is the horizontal velocity vector averaged over the layer depth H, and k is the unit vector along the Earth rotation axis. For simplicity, the Coriolis parameter is assumed to be constant (f-plane approximation) and the gravity force g parallel to k. Therefore, our consideration applies to mid and high latitudes and to spatial scales not exceeding a few hundred kilometers. (In the rest of this paper, symbol k is employed for a different purpose: it designates the wavenumber vector. We hope this will not cause any confusion.)

For a baroclinic case, the discussion will be conducted in terms of a 2-layer model, hence we limit our consideration to the 1st baroclinic mode. The g in (3) is then to be viewed as the reduced gravity and $H=H_1H_2/(H_1+H_2)$, where subscripts 1 and 2 correspond to the upper and lower layers, respective y, and $H_1+\zeta(\mathbf{x},t)$ is the interface between the two layers (the thermocline depth). We also assume $H_1 \ll H_2$ and H_1 to be sufficiently large compared to the amplitude of $\zeta(\mathbf{x},t)$ oscillations (e.g., [LeBlond and Mysak, 1978; Gill, 1982]). Under these conditions, the free surface plays a passive role: its undulations mimic oscillations of the thermocline boundary - although with a much smaller amplitude.

We are interested in stationary wave solutions of (3) whose general form can be written as $\zeta(\mathbf{x},t) = \int e^{i(\mathbf{k}\cdot\mathbf{x}+\omega(\mathbf{k})t)} dZ(\mathbf{k})$ with $dZ(\mathbf{k})$ representing the complex amplitude of the surface height (or thermocline boundary) spatial variations. A similar expression can be written for the velocity field. The presence of advective terms in (3) leads to the energy exchange among Fourier components. As a result, the wave field exhibits a highly complicated (random) behavior. An appropriate description of such fields is provided by their statistical moments. The simplest such characteristic is the power spectrum $F_\zeta(\mathbf{k})$ of $\zeta(\mathbf{x},t)$ variations

$$F_\zeta(\mathbf{k}_1)\delta(\mathbf{k}_1-\mathbf{k})d\mathbf{k}_1 = \left\langle dZ(\mathbf{k})dZ^*(\mathbf{k}_1)\right\rangle \tag{4}$$

where the angle brackets denote ensemble averaging. This implies statistical spatial homogeneity of field $\zeta(\mathbf{x},t)$. Subscript ζ becomes useful in section 3 - when we introduce energy spectra.

A statistically stationary wave field is observed if the external source, supplying energy at a constant rate, works long enough to result in a steady spectral flux of wave energy (and of other quantities conserved in the spectral cascade). In many problems of wave turbulence the external input is considered to be confined to a limited band of wavenumbers/frequencies. In other words, outside the "generation range" the spectral fluxes are assumed to be purely inertial. Similarities with the 3- or 2-D eddy turbulence in an incompressible fluid are obvious. For instance, the energy flux toward high wavenumbers is identified with the rate of energy dissipation (as in 3-d turbulence) and an inverse spectral cascade is possible (as in 2-d turbulence) [Zakharov, 1984]. There are also considerable differences between eddy and wave turbulence. In particular, wave turbulence depends on the intrinsic relationship between wavenumber k and frequency ω. For eddy turbulence, dispersion relationships do not exist. Similar to the case of surface gravity waves on deep water, (1) forbids 3-wave resonant interactions. The lowest-order resonance occurs in wave tetrads

$$\begin{aligned} &\mathbf{k}_1 \pm \mathbf{k}_2 \pm \mathbf{k}_3 \pm \mathbf{k}_4 = 0 \\ &\omega_0 \pm \omega_2 \pm \omega_3 \pm \omega_4 = 0 \end{aligned} \tag{5}$$

The formal, small-perturbation theory describes this type of processes in the framework of the kinetic equation [Hasselmann, 1962; Zakharov et al., 1992] which is usually written for the spectral density of wave action, $N(\mathbf{k},t) = F(\mathbf{k},t)/\omega$, where $F(\mathbf{k},t)$ is the energy spectrum. The relationship between $F(\mathbf{k},t)$ and $F_\zeta(\mathbf{k},t)$ may be rather complicated. For the inertial interval, the kinetic equation in the approximation of 4-wave interactions is

$$\partial N(\mathbf{k},t) / \partial t = \int |T_{0123}|^2 \delta(\omega + \omega_1 - \omega_2 - \omega_3)\delta(\mathbf{k} + \mathrm{k}_1 - \mathrm{k}_2, - \mathbf{k}_3) f_{k123} d\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \tag{6}$$

in which $f_{k123} = N_k N_1 N_2 N_3 (1 / N_k + 1/N_1 - 1/N_2 - 1/N_3)$ and. the subscripts designate arguments (such ask, k $_1$, etc.). The integral describes "collisions" which result in the "birth" of two waves in place of two initial waves. Other types of collisions are neglected. The scattering matrix T_{k123} is a complicated function of wavenumber vectors and frequencies. Its explicit expression for IG waves is given by Falkovich and Medvedev [1992].

For our subsequent development it is important to note that the integral in (6) represents the divergence (in the wavenumber space) of the spectral flux, P, of wave action. To emphasize this point, the collision integral can be denoted by $\nabla_k \cdot P(\mathbf{k}, t)$ [Phillips, 1977]. In a steady state, (6) takes a simple form

$$\nabla_k \cdot \mathrm{P(k)} = \mathrm{O} \tag{7}$$

Its l-d version - obtained by integrating (7) as $\int_{-\pi}^{\pi} (\ldots) k d\theta$ - gives the conservation of the wave action flux in the spectral cascade, $\partial P(k) / \partial k$ = O, or:

$$\mathrm{P(k)} = \mathbf{P}_0 \tag{8}$$

A similar expression is obtained for the energy flux -by multiplying (7) by $\mathbf{k}\omega(\mathbf{k})$ and integrating over azimuthal directions. The result is $\partial Q(k) / \partial k$ = O or:

$$Q(k) = Q_0 \tag{9}$$

where Q_0 is the rate of energy dissipation. Equation (8) may have several solutions [Zakharov, 1984] of which only one or two have physical significance. In particular, this is the case for weakly nonlinear, sufficiently short IG waves [Falkovich and Medvedev, 1992] for which (1) can be replaced by

$$\omega \cong C_{0,m} k \left(1 + \frac{1}{2}(kR)^{-2}\right) \tag{1o}$$

Two special solutions of(8) correspond to two types of spectral cascade: the inverse cascade of wave action and the direct cascade of wave energy [e.g., Zakharov et al., 1992;

Falkovich and Medvedev, 1992]. In what follows, wc assume that both cascades are local in the wavenumber space. Implications and validity of this assumption with respect to the direct energy cascade are discussed in sections 3 and 5.

So far, equations of type (8) yielded closed-form solutions only for scale-invariant systems characterized by simple dispersion laws and simple expressions for the wave energy. Scale invariance leads to drastic simplifications of the scattering matrix T_{k123} which allow one to seek solutions in the form $N(k) \propto k{-r}$. For short (but not too short) waves, an approximate solution corresponding to the inverse cascade is

$$F(k) \propto P^{1/3} k^{-10/3} \tag{11}$$

[Falkovich and Medvedev, 1992]. This is a 2.-d spectrum of wave energy in which the angular dependence is omitted (being assumed constant). Solution (11) is valid only for a very narrow range of wavenumbers in which the reduced dispersion (10) is as weak as the wave nonlinearity. In section *3, we* provide a more general result for the inverse cascade.

When the nonlinearity is greater than that implied in the lowest-order theory, the derivation of a kinetic equation (to account for higher-order terms in the perturbation series) becomes impractical. However, as an introduction to the heuristic arguments of section 3, it is useful to review some qualitative aspects of the formal perturbation theory. Suppose the perturbation expansion is carried to an arbitrary order, The first nonlinear terms in deterministic equations (for properly normalized and partially-time-averaged Fourier amplitudes $a(\mathbf{k},t) \propto \varepsilon$), are of order ε^2 and the subsequent terms are of order $\varepsilon^3, \varepsilon^4$, etc., where ε is the measure of nonlinearity. For most wave problems, including IG waves, ε = u/c, where u is the characteristic velocity of water particles and c is the wave phase speed (for a given wavelength). Suppose further that, based on the deterministic equations, one can derive a closed-form kinetic equation for statistical moments such as $\langle a(\mathbf{k},t)a^*(\mathbf{k}_1,t)\rangle$ where the angle brackets denote ensemble averaging. This equation would have the form

$$\frac{\partial N(\mathbf{k};t)}{at} + \nabla_k P^{(3)} + \nabla_k P^{(4)} + \nabla_k P^{(5)} + \ldots = \gamma(\mathbf{k}) N(\mathbf{k},t) \tag{12}$$

where "partial" collision integrals $\nabla_k P^{(n)}$ account for n-wave interactions, If 3-wave interactions are non-re sonant, term $\nabla_k P^{(3)}$ is eliminated by an appropriate canonical transformation of the Hamiltonian. This, however, is relevant only to weakly non-linear waves [Zakharov et al., 1992]. Apparently, the collision integrals in (12) scale as $\varepsilon^{2(n-1)}$. Furthermore, $\gamma(\mathbf{k})N(\mathbf{k},t)$ represents the external source (or sink, or both) where $\gamma(\mathbf{k})$ is an increment (decrement) of wave growth (attenuation). As mentioned earlier, in the inertial range we have $\gamma(\mathbf{k})=0$. Weakly nonlinear waves (i.e., $\varepsilon << 1$) permit neglecting all partial collision integrals except for the first one. Indeed, if& ≈ 0.1, the first collision term in (12) is 102 times as large as the subsequent terms. However, this is not the case if the nonlinearity is stronger. For a weak inequality&< 1, we would have to retain a series of terms (up to n ≈ 6 for the case of $\varepsilon \approx 0.5$) in order to maintain the same accuracy as in our example with $\varepsilon \approx 0.1$. Of course, the numbers should not be taken too literally, for the real situation is more complicated. However, the suggestion that multi wave interactions with n=5, 6, etc., could become as important as the lower-order interactions appears plausible. Finally when $\varepsilon \rightarrow 1$, interactions of all orders up ton $\rightarrow \infty$ become of comparable importance. This case of strong wave turbulence results in "saturated" spectra. Larraza et al. [1989] showed that the Phillips spectrum $F(k) \sim k^{-4}$ for deep-water gravity waves is just one example. A similar consideration is presented in [Glazman, 1993] for capillary waves, while a monotonous transition from weak to strong turbulence is studied in [Glazman, 1992] for deep-water gravity waves. Theoretical and empirical relationships between the effective number of resonantly interacting wave components and the external parameters of the problem are suggested in [Glazman, 1992, 1995.

3. Heuristic theory of IG wave turbulence

Let us introduce the highest relevant number, v, of resonant wave-wave interactions on the hypothesis that all interactions up to this order make comparable contributions to the spectral flux. Symbolically this can be written in the form

$$\frac{\partial N(\mathbf{k};t)}{\partial t}+\sum_{n=3}^{\nu}\nabla_k P^{(n)} = \gamma(\mathbf{k})N(\mathbf{k},t) \qquad (13)$$

For the inertial range in a steady state we have (8) with $P=\sum_{n}^{\nu}P^{(n)}$ or *(9)* with

$$Q(k)=\sum_{n}^{\nu}\omega(k)P(k)^{(n)} \qquad (14)$$

The "effective number" ν of resonant wave components represents a measure of the wave nonlinearity, hence it is an increasing function of the "internal parameter" ε and can be, hopefully, related to external factors, such as the energy input Q_0.

Considering the energy cascade, one can introduce the amount of energy, E_j, transferred by the local nonlinear interactions from a given narrow range of scales (the j-th step in the cascade) to the next (the j+1 step in the cascade):

$$E_j=\int_{k_j}^{k_{j+1}}F(k)k\,dk\ , \text{ where } k_{j+1}/k_j=r \qquad (15)$$

and $r>1$ is a constant. The characteristic time t_j of the spectral transfer at step j, called the turnover time, should be taken as the largest among all individual turnover times associated with partial fluxes in (13). Indeed, it is the slowest process that controls the total flux. By scaling the terms in (13), one ultimately finds

$$t_j^{-1}\sim\omega\varepsilon^{2(\nu-2)} \qquad (16)$$

where ω is related to k by (1) and $\varepsilon\equiv u/c_{0,m}$ is explained in the preceding section.

Expression (16), with $\nu=4$ and $\varepsilon=ak$ where *a is the* characteristic wave amplitude, was employed by Kitaigorodskii [1983] and Phillips [1985] tore-derive the Zakharov-Filonenko [1966] spectrum of weak turbulence in deep-water gravity waves. For an arbitrary value of ν, (16) was suggested by Larraza et al. [1989] who showed that the Phillips [1958] spectrum is obtained as a result of an increasing interaction time when $ak\rightarrow 1$ (thus causing the number of resonant wave-wave interactions to become infinite). Obviously, this cascade model of wave turbulence is rather similar to that of eddy turbulence [e.g., Frisch et al, 1978].

In terms of (15) and (16), the spectral flux of wave energy is simply

$$Q = E_j / t_j \tag{17}$$

Equating this to the input flux, Q_0, we shall use $E_j / t_j = Q_0$ in place of the kinetic equation. Relating E_j and t_j to k, ω and other relevant quantities one can estimate the shape of the spectrum. his, however, presumes that the spectrum falls off sufficiently fast as the wavenumber increases [Glazman, 1995]. Indeed, differentiating (15), $\partial E_j / \partial k_j = -(F(k_j)k_j - F(k_j r)k_j r^2)$, *we* find that

$$F(k_j) \approx -\frac{1}{k_j}\frac{\partial E_j}{\partial k_j} \tag{18}$$

if, and only if, the spectrum falls-off not slower than k^{-s} where the exponents satisfies $r^{2-s} << 1$. Assuming for now that this condition holds, we shall check it *a posteriori.* (In what follows, we will often omit the subscript j at k and ω.)

The kinetic and potential energies of IG waves (per unit surface area, per unit mass of water) are

$$EK = \frac{\langle |\mathbf{U}|^2 \rangle}{2}, \quad EP = \frac{g\langle \zeta^2 \rangle}{2H}. \tag{19}$$

(When averaging over the surface area, we assumed periodic boundary conditions.) For a narrow frequency band between k_j and k_{j+1}, these energies are referenced to the corresponding characteristic scales of the wave amplitude, a_j, wave number, k_j, etc. The ratio of these energies increases with an increasing wavelength (e.g., Gill, 1982]). Approximately,

$$\frac{EK_j}{EP_j} \approx 1 + \frac{2}{(kR)^2} \tag{20}$$

Physically, the absence of energy equi-partition is due to the fact that the orbits of water particles are not strictly vertical (as would be the case for pure gravity waves). Their inclination is the greater, the larger the relative importance of the Coriolis force (e.g., [LeBlond and Mysak, 1978; Fig. 17.1]). Since the total energy is E=EK+EP, it is useful to express both components in terms of *Ej:*

$$EK_j \approx \frac{E_j}{2}\left(1 + \frac{1}{1+(kR)^2}\right) \tag{21.1}$$

$$EP_j \approx \frac{E_j}{2} \frac{(kR)^2}{1+(kR)^2} \tag{21.2}$$

In viewof(19), the characteristic particle velocity at scale k *is given* by

$$u^2(k) \approx E_j\left(1+\frac{1}{1+(kR)^2}\right) \tag{22}$$

Based on(1) the characteristic phase speed is

$$c^2(k) \approx C_0^2\left[1+\frac{1}{(kR)^2}\right) \tag{23}$$

We can now express the interaction time, t_j, in terms of E_j, k, $\omega(k)$ and C_0 (which is the same as $C_{0,m}$ appearing in (l), (2) and (10)):

$$t_j^{-1} \approx C_0 R^{-1}\left(1+\tilde{k}/2\right)\left[\frac{E_j}{C_0^2}\left(1-\frac{1}{(1+\tilde{k}^2)^2}\right)\right]^{\nu-2} \tag{24}$$

where $$\tilde{k} = kR \tag{25}$$

is the non-dimensional wavenumber. It is also convenient to non-dimensionalize other quantities:

$$\tilde{Q}_0 = Q_0(R/C_0^3), \qquad \tilde{E}_j = E_j/C_0^2, \qquad \tilde{F}(\tilde{k}) = F(k)/(C_0R)^2 \tag{26}$$

With t_j^{-1} given by (24), equation (17) can be solved for E_j. In the non-dimensional form, the result is

$$\tilde{E}_j = \tilde{Q}_0^{1/(\nu-1)} z^{-1/2(\nu-1)}(1-1/z^2)^{-(\nu-2)/(\nu-1)} \tag{27}$$

where we introduced

$$z = 1+\tilde{k}^2 \tag{28}$$

An additional advantage of using the scaled variables is that this enables us to replace sign "≈" in our relationships with "=" - implying that an appropriate constant of proportionality (playing the role of the Kolmogorov constant) has been incorporated into Q_0 of (26). In terms of z of (28), equation (1 8) takes the form

$$\tilde{F}(\tilde{k}) \approx -2\left.\frac{\partial \tilde{E}_j}{\partial z}\right|_{z=1+\tilde{k}^2} \tag{29}$$

This yields the 2-d spectrum of the total wave energy:

$$\tilde{F}(\tilde{k}) = \frac{\tilde{Q}_0{}^{1/(\nu-1)}}{(\text{v-1})} z^{(\nu-7/2)/(\nu-1)} (z^2-1)^{-(2\nu-3)/(\nu-1)} (z^2+4\nu-\ 9) \tag{30}$$

The surface height spectrum (i.e., the potential energy spectrum) is found based on (21 .2):

$$\tilde{F}_\zeta(\tilde{k}) \cdot \frac{\tilde{Q}^{1/(\nu-1)}}{(\text{v-1})} \frac{(\text{z - 1})(22+\ 4\text{V - 9})}{(z^2\ \text{-1})^{(2\nu-3)/(\nu-1)} z^{5/2(\nu-1)}} \tag{31}$$

Since the angular dependence in our 2-d spectra is forgone, the corresponding l-d spectrum is simply $\tilde{F}_\zeta(\tilde{k})\tilde{k}$. The plot of this spectrum is shown in Fig. 1 for several values of v.

For weakly non-linear waves (when v $\approx$ 4), an additional physically-meaningful solution of(7) corresponds to the inverse cascade of wave action. This solution is obtained by considering the action flux, i.e.

$$P = N_j / t_j \tag{32}$$

where $N_j = \int_{k_n}^{k_{n+1}} F(k)\omega^{-1} k dk$. In the same fashion as before, we arrive at:

$$\tilde{F}(\tilde{k}) \approx \frac{2\tilde{P}_0{}^{1/3}}{3} z^{1/3} (z^2-1)^{-5/3} \tag{33}$$

The corresponding spectrum of surface height variations is

$$\tilde{F}_\zeta(\tilde{k}) \approx \frac{2\tilde{P}_0{}^{1/3}}{3} z^{-2/3} (z^2-1)^{-5/3} (z-1) \tag{34}$$

Figure 2 provides comparison of the energy spectrum (33) with the spectrum(11) derived based on a reduced wave dispersion (10). Again, we plot the results in terms of 1-d spectra $\tilde{k}F(\tilde{k})$. The figure gives an idea about the range in which (11) is in an approximate agreement with our prediction: roughly this is $kR < 2$. At higher wavenumbers, when the wave dispersion becomes small compared to the wave nonlinearity, the Falkovich-Medvedev theory does not apply,

The spectra in Fig. 1 contain interesting information. The first (actually, very smooth) ''break" in the spectrum (at $kR \approx 0.8$) tentatively separates the range of inertial oscillations from that strongly affected by the gravity force. The second (also rather smooth) "break" (at $\tilde{k} \approx 3$) occurs only if the wave nonlinearity is sufficiently high (v $\geq$

8). Thus, in the range 0.8< $\tilde{k} < 3$ we have "fully" dispersive IG waves whose spectrum is sensitive to the degree of the wave nonlinearity. In this range the spectrum fall-off may be as fast as k^{-3} - greater than in either of the asymptotic regimes of $\tilde{k} \to 0$ and $\tilde{k} \to \infty$. This is the most non-trivial result of the present them-y. At $\tilde{k} > 3$, wave dispersion has only a weak effect on the spectrum (unless the nonlinearity is extremely high). Generally, the greater the wave nonlinearity, the higher the wavenumbers at which wave dispersion influences the spectrum. This is evidenced in a shift of the second spectral break toward higher $\tilde{k}$ as v increases.

Finally, Fig. 1 shows that the regime of non-dispersive ("acoustic") waves observed at $\tilde{k} \gg 1$ is characterized by very flat spectra (approaching - k^{-1} in terms of the 1-d spectrum $F(k)k$). The nonlinearity of such waves (no matter how weak) is not counteracted by competing factors, hence it should eventually lead to the formation of "shock" waves [Whitham, 1974] accompanied by a vigorous wave breaking. Indeed, spectrum F(k) - k^{-1} means that the (1-d) surface $\zeta(x)$ is discontinuous in the mean-square sense. Hence, at small scales, wave crests tend to become highly asymmetric, having a steep front face and a gentle rear face. In section 5 we discuss imp] ications of this spectral range for dynamical processes in the thermocline.

The small spectral slope at high wavenumbers has the following implications for the validity of (18). The short-wave asymptotic of (30) and (31) are given by

$$\tilde{F}(\tilde{k}) \propto \tilde{k}^{-s}\left(1 + 4\nu\tilde{k}^{-4}\right) \tag{35}$$

where $s = 2 + 1/(\nu - 1)$. Apparently, the necessary condition for(18) to be valid is: s > 2. A strong inequality, s >> 2, means that the spectral cascade is local. This is so because the spectral width of a cascade step (measured by the rof(15)) would not have to be very large to satisfy the requirement r^{2-s} <<1 underlying (18). At high $\tilde{k}$ - as follows from (35) - the locality of wave-wave interactions comes into question because we only have a week inequalitys > 2. Therefore, in the high-wavenumber range our simple theory needs to be complemented with a model of energy dissipation due to an intermittent breaking of steep wavelets. Such a model would describe a non-local energy transfer to small scales, similar to that observed in deep-water gravity waves.

4. Analysis of Sea Surface Height Spectra

To some extent, the present work was inspired by satellite altimeter observations showing that power spectra of the SSH field often exhibit features of scale-dependent wave turbulence. However, the IG wave turbulence represents only one, not necessarily dominant, component of the total SSH variability. Filtering out fast SSH oscillations (with periods comparable to that of baroclinic IG waves), this component disappears [Glazman et al., 1995], In the wavenumber range of our interest, the spectra of slow motions are dominated by the 2-d eddy turbulence. (At lower wavenumbers, we also observe a contribution of baroclinic Rossby waves [Glazman et al., 1995]).

One-dimensional spectra of SSH spatial variations are presently well documented (e.g., [Fu, 1983], [De Mey and Menard, 1989], [Le Traon et al, 1990; 1994]), and 2-d spectra and spatio-temporal autocorrelation functions are beginning to be reported [Glazman, et al., 1995]. All such spectra are based on SSH data from which time-invariant spatial trends have been removed. Typical spectra of these "de-trended" SSH variations along satellite passes are presented in Fig.3, reproduced from [Le Traon et al., 1994]. The spectra are calculated separately for ascending and descending passes and then averaged together. Therefore, they utilize practically instantaneous measurements because it takes only about 3 min for a satellite to cover a 1000 km groundtrack segment. At wavenumbers above the first spectral break, these spectra appear to be in good agreement with the theoretical prediction (31) based on the direct energy cascade. The spectrum (34) based on the inverse cascade displays agreement with the observations at wavenumbers roughly below $\tilde{k} = 2$ -as shown by Fig. 2 curve (b).

An explanation of the observed trends is suggested based on the following view. As is well known, most of the mesoscale eddy energy is contained at wavenumbers near 0.02 rad/km (i.e., at scales near *300 km).* It is also likely that most of the baroclinic IG wave energy is generated in this particular range: hydrodynamic instability of a 2-d vertical flow with respect to baroclinic gravity modes could provide the energy source. Assuming the energy input to be at maximum at wavenumbers near 0.02 rad/km, one anticipates (by analogy with the wind-generated surface gravity waves on deep-water) the existence of two spectral cascades of IG wave turbulence: the direct cascade in the higher-wavenumber range and the inverse cascade at lower wavenumbers - away from the generation range.

The characteristic period of baroclinic IG waves, $2\pi / \omega$, is estimated based on (1) where $C_{0,1} \approx 3$ m/s (a typical baroclinic Kelvin wave speed for the first mode in a mid-latitude region). The corresponding Rossby radius is about 30 km. It is easy to check that, for the wavelength range 10 to 300 km, the wave period varies between 1 and 15 hours. Recalling (25) one can also see that the position of the spectral breaks in Fig. 3 is in reasonable agreement with those in Figs. 1 and 2 (curve b).

It is well known that the 1-d spectra based on along-track measurements exhibit a rather gentle spectral fall-off (k^{-3} and slower) only in the low-energy regions (e.g., [Fu, 1983], [Le Traon et al., 1990]). Near the Gulf Stream, for example, the spectra fall off at least as fast as k^{-4}. An explanation of this behavior is simple: in the regions of high meso-scale eddy activity, the SSH variations are dominated by the vertical rather than gravity-wave component. The kinetic energy spectra of 2-d eddy turbulence are given by k^{-3} or $k^{-5/3}$, corresponding to the enstrophy or energy cascades, respective] y [Kraichnan, 1967], In terms of l-d spectra of SSH variations, these translate into k^{-5} or $k^{-11/3}$. Such high rates of spectral decay are consistently observed in high-energy regions [Fu, 1983], [Le Traon, et al., 1990], [Glazman, et al., 1995].

Apparently, the strong dependence of the (dimensional) wave spectrum on the internal Rossby radius of deformation points to a possibility of inferring this oceanographic parameter from spectra of SSH variations along altimeter groundtracks. Moreover, the spectral “breaks” seen in Fig. 3 (actually, the entire shape of the spectrum) allow one to estimate the degree of the nonlinearity of baroclinic waves - an important indication of dynamical activity in the thermocline.

Comparing different curves in Fig. 1 to the spectra in Fig. 3 and to spectra reported for various regions by Le Traon et al. [1990], one concludes that the degree of the wave nonlinearity (in terms of parameter v) varies from region to region, but is typically quite high. This should cause no surprise. Really, due to the smallness of the thermocline depth, the relative wave amplitude (or, equivalently, the ratio of the particle velocity to the phase speed) can be rather large. The physical cause of large-amplitude internal waves is the extremely small difference in the densities of the two layers. This allows large oscillations to be produced by low-energy forcing. The barotropic waves whose characteristic phase speed is about 200 m/s could hardly ever attain a similar degree of nonlinearity.

5. Discussion and conclusions

The present analysis addressed only the most basic aspects of the problem. By limiting our consideration to a single (first) baroclinic mode we excluded possible exchange of energy with other modes. This exchange may be important, at least within a certain subrange of wavenumbers/frequencies. Further theoretical developments should also include a more elaborate treatment of the Coriolis force (which becomes important for the regions closer to the equator). Furthermore, since the spectral distribution of energy input to IG wave turbulence is not yet known, our consideration has been limited to purely inertial cascades - when all the input is concentrated at unspecified low or high wavenumbers. In principle, one can generalize the theory to account for a continuos spectral distribution of the energy input. Such a generalization would also reduce the uncertainty in the effective value of v (the number of the resonantly interacting wave components) [Glazman, 1992]. However, we do not yet have enough knowledge about relevant physical processes to address such issues.

Experimental comparisons indicate that the IG wave spectra at higher wavenumbers are determined by the direct energy flux. This has interesting implications. Specifically, spectral transfer of energy from larger scales should result in energy dissipation at high wavenumbers. This dissipation can occur through a variety of mechanisms of which the internal wave breaking and mixing are most likely. Wave breaking produces small scale turbulence. For wind-generated surface gravity waves, the probability and other statistics of the wave breaking have been related to the spectrum shape [Snyder and Kennedy, 1983], [Glazman and Weichman, 1989]. Therefore, we anticipate that further studies of IG wave turbulence may lead to quantitative characterization of intermittent events of baroclinic wave breaking. Such studies might improve. our understanding of vertical fluxes and ocean mixing processes.

The theory indicates that the thermocline oscillations contain important oceanographic information which can be derived from observed SSH spectra. The Rossby radius of deformation is one such item. Another, presently less understood, information item is the degree of the wave nonlinearity. This is inferred by analyzing the spectral slope in different wavenumber subranges: according to Fig. 1, a sharp change in the spectral slope points to a high degree of nonlinearity, whereas a near -constant slope (the case of $v = 4$ in Fig. 1) corresponds to the lowest degree of nonlinearity.

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Figure Captions.

Figure 1. The 1-d wavenumber spectrum $\tilde{F}_{\zeta}(\tilde{k})\tilde{k}$ of SSH spatial variations predicted based on (31). The degree of the wave nonlinearity, in terms of the effective number v of the resonantly interacting Fourier components, is indicated at each curve.

Figure 2. Solid curves: 1-d spectra of wave energy (a) and of interface spatial variations (b) due to weakly-nonlinear IG wave turbulence (v =4) for the inverse cascade of wave action. The spectra are based on (34) and (33), respectively. Dashed curve: asymptotic solution (11) [Falkovich and Medvedev, 1992] plotted as a 1-d spectrum $k^{-7/3}$.

Figure 3. 1-d spectra of spatial SSH variations along altimeter groundtracks [LeTraon et al., 1994] based on Topex/Poseidon data for mid-latit ude regions. (Permission to reproduce this figure will be requested from the authors upon successful review of the manuscript.)

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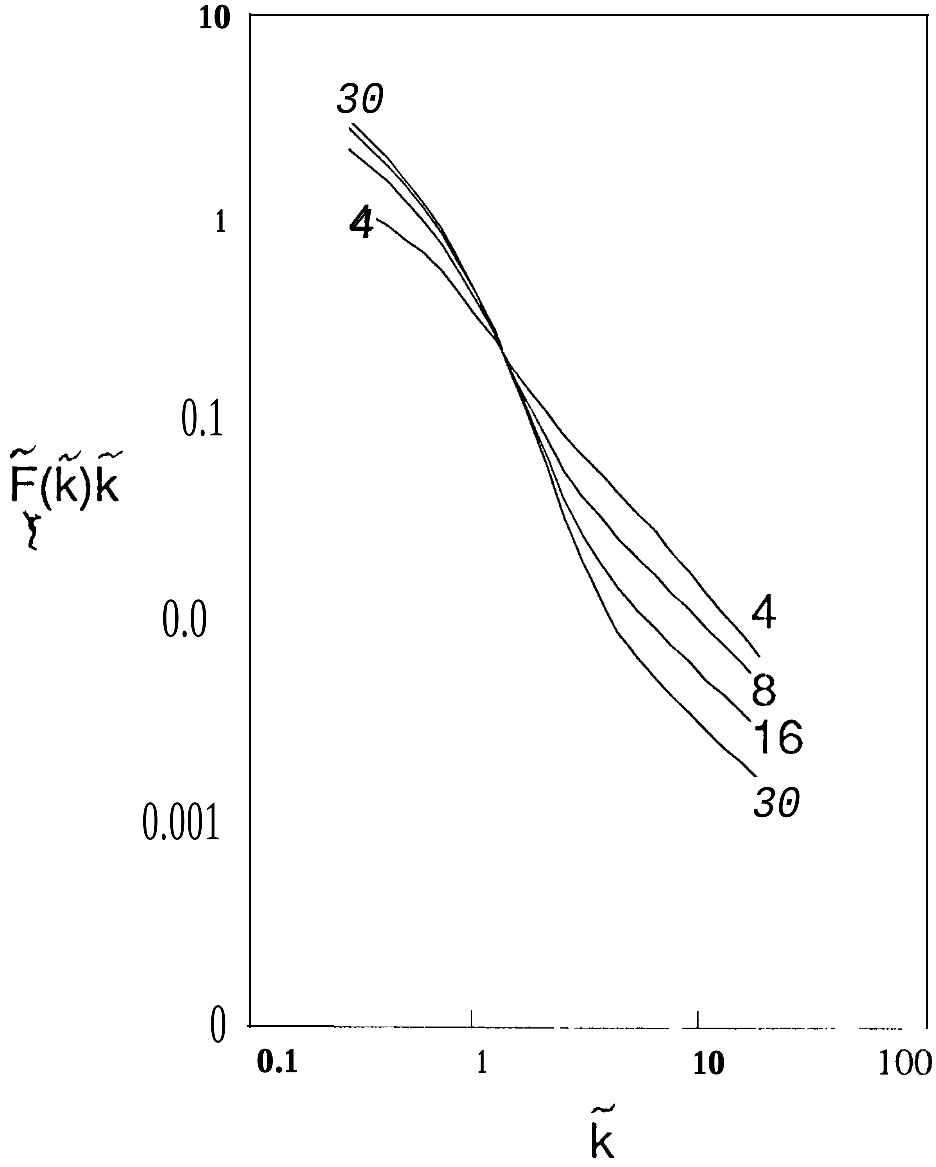
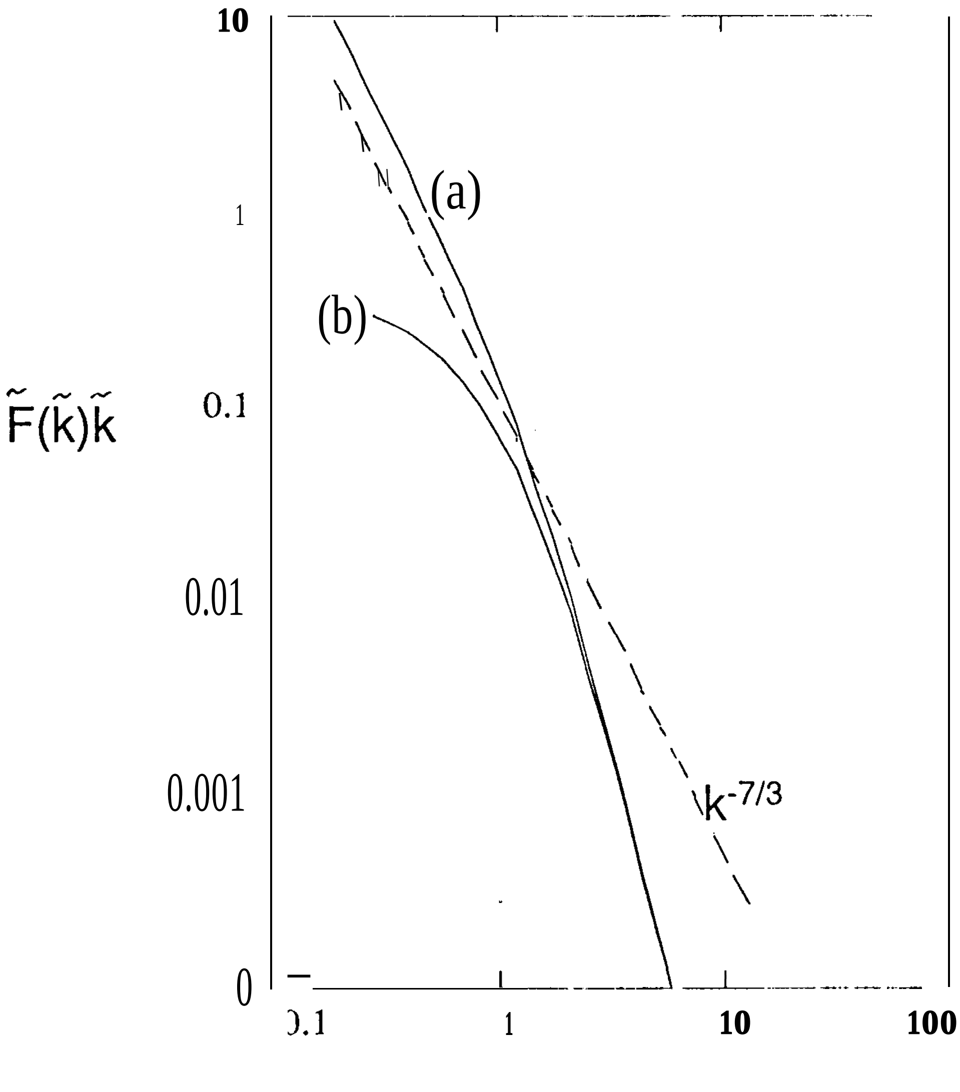

10
1
0.1
0.0
0.001
0
0.1
1
10
100
$\tilde{F}(\tilde{k})\tilde{k}$
$\tilde{k}$
30
4
4
8
16
30

Fig. 1



10
1
0.1
0.01
0.001
0
$\tilde{F}(\tilde{k})\tilde{k}$
(a)
(b)
$k^{-7/3}$
0.1
1
10
100
$\tilde{k}$

Fig. 2

Fig. 3

1-D spectra of sea surface height along satellite passes [Le Traon et al., 1994]